Generalized Third-price Auctions

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1 Preliminaries

1.1 The Environment

We first describe the environment of sponsored search auctions that was introduced in Edelman et al. (2007) and Varian (2007). For a given keyword, let $N = \{1, 2, \ldots, n\}$ be the set of agents (advertisers) who are interested in displaying their ads on the result page; and advertiser i's maximum willingness to pay for a click is denoted by v_i. The number of ads that the search engine can display per search is limited to k; and let c_j denote the expected number of clicks that the ad in position j receives per period of time. If advertiser i buys position j for price p_j per click, then his profit is simply $c_j(v_i - p_j)$. Without loss of generality, advertisers and positions are labeled in descending order, i.e., $v_1 > v_2 > \ldots > v_n > 0$ and $c_1 > c_2 > \ldots > c_k > 0$.

It is usually the case that the number of advertisers is much larger than the number of positions since only limited number of ads can be displayed per result page, we therefore assume $n \geq k + 2$ throughout. To make the notations simpler, we equate the number of positions to the number of advertisers by adding $n - k$ fictitious positions with zero click-through-rate each. That is, $c_j = 0$ for all $j = k + 1, \ldots, n$. We use N as the set of positions for convenience.

An auction mechanism M in this setting specifies a pair of allocation $\pi : N \to N$ and payment vector $p \in \mathbb{R}^n$ for each bid profile $b = (b_1, b_2, \ldots, b_n)$ submitted by the advertisers, where $\pi(i)$ is the identity of the advertiser who wins position i for price p_i. Advertiser $\pi(i)$'s total payment P_i is $c_i p_i$, so the total revenue of the seller is then $R^M(b) = \sum_{i=1}^n P_i$. The social welfare is $W^M(b) = \sum_{i=1}^n c_i v_{\pi(i)}$. An allocation is called *efficient* if it maximizes the social welfare, i.e., $\pi(i) = i$ for all $i \leq k$.

VCG mechanism allocates the first position to the advertiser with the highest bid, and the second position to the advertiser with the second-highest bid, and so on; and it charges each advertiser the externality he imposes on others. In the game induced by VCG mechanism, it is optimal (dominant strategy) for each advertiser to bid his true valuation. In this truthful equilibrium, the allocation is efficient and the total payment for each position is calculated as following:

$$\begin{aligned} P_k^{VCG} &= c_k v_{k+1} \\ P_i^{VCG} &= (c_i - c_{i+1}) v_{i+1} + P_{i+1}^{VCG}, \quad \forall i < k \end{aligned} \tag{1}$$

We take the truthful VCG outcome as a benchmark to compare the performance of other auction mechanisms in terms of the efficiency and the seller's revenue. Suppose a bid profile b in auction M generates an efficient allocation, and the payment for each position satisfies the following:

$$P_k^M \geq c_k v_{k+1} \tag{R1}$$

$$P_i^M \geq (c_i - c_{i+1}) v_{i+1} + P_{i+1}^M, \quad \forall i < k \tag{R2}$$

Then, clearly, by induction, we get $P_i^M \geq P_i^{VCG}$ for all $i \leq k$; and therefore the seller's revenue in M, $R^M(b)$, would be at least as high as the VCG revenue, R^{VCG}.

1.2 Generalized Second-Price Auction

In GSP, the advertiser with the highest bid wins the first position, but pays the second-highest bid per click; the advertiser with the second-highest bid wins the second position, but pays the third-highest bid per click; and so on. Ties are broken uniformly at random. Given a bid profile $(b_1, b_2, \ldots, b_n)$, the payoff of advertiser $\pi(i)$, who wins position i, is simply $c_i \langle v_{\pi(i)} - b_{\pi(i+1)} \rangle$, where $\pi(n+1) = n+1$ and $b_{n+1} = 0$. We denote this game by Γ^{GSP}.

A bid profile $(b_1, b_2, \ldots, b_n)$ is a Nash equilibrium of Γ^{GSP}, if:

$$\begin{aligned} c_i \langle v_{\pi(i)} - b_{\pi(i+1)} \rangle &\geq c_j \langle v_{\pi(i)} - b_{\pi(j+1)} \rangle, \quad \forall i, j \in N : j \geq i \\ c_i \langle v_{\pi(i)} - b_{\pi(i+1)} \rangle &\geq c_j \langle v_{\pi(i)} - b_{\pi(j)} \rangle, \quad \forall i, j \in N : j < i \end{aligned} \tag{2}$$

Indeed, if advertiser $\pi(i)$ who wins position i wants to move down to position j ($j > i$), he would bid just above the bid of advertiser $\pi(j+1)$ and pays $b_{\pi(j+1)}$. To avoid such deviation, the first condition in (2) is imposed. On the other hand, to move up to position j ($j < i$), advertiser $\pi(i)$ would have to bid at least the bid of advertiser $\pi(j)$ and would pay $b_{\pi(j)}$, hence the second condition is imposed to avoid such deviation.

It turns out that GSP game has many Nash equilibria, involving the ones with inefficient outcome, or low payment, or both. Edelman et al. (2007) and Varian (2007) select a specific subset of Nash equilibria by imposing further conditions.

Definition 1 (Edelman et al, 2007)**.** *A Nash equilibrium* $(b_1, b_2, \ldots, b_n)$ *of game* Γ^{GSP} *is locally envy-free, if*

$$c_i \left\langle v_{\pi(i)} - b_{\pi(i+1)} \right\rangle \geq c_{i-1} \left\langle v_{\pi(i)} - b_{\pi(i)} \right\rangle, \quad \forall i \in N \tag{3}$$

Edelman et al. (2007) shows that the outcome of any locally envy-free equilibrium in Γ^{GSP} is efficient, and the total revenue of the seller in it is at least as high as in the truthful VCG outcome. However, in the next section, we show an example that contradicts these results.

2 On the Locally Envy-free Condition

Example. Suppose there are 4 advertisers competing for 2 positions. Position 1 gets 200 clicks per period, and Position 2 gets 100. Advertisers 1, 2, 3, and 4 have values 10, 5, 4, and 2 per click, respectively. One can easily verify that a bidding profile $b = (10, 1, 5, 2)$ constitutes a locally envy-free equilibrium of the GSP game. However, this equilibrium doesn't yield an efficient outcome as advertiser 3 wins position 2. Moreover, the total revenue of the seller in this equilibrium is $c_1 b_3 + c_2 b_4 = 1200$ whereas VCG revenue is 1300.

Let us first see what went wrong in their analysis. Let $(b_1, b_2, \ldots, b_n)$ be a locally envy-free equilibrium of Γ^{GSP}. Then, by the equilibrium condition, advertiser $\pi(i)$ is not better of undercutting the bid of advertiser $\pi(i+1)$, that is:

$$c_i \left\langle v_{\pi(i)} - b_{\pi(i+1)} \right\rangle \geq c_{i+1} \left\langle v_{\pi(i)} - b_{\pi(i+2)} \right\rangle$$

Moreover, by the locally envy-freeness condition, advertiser $\pi(i+1)$ is not better off by exchanging bids with advertiser $\pi(i)$, that is:

$$c_{i+1} \left\langle v_{\pi(i+1)} - b_{\pi(i+2)} \right\rangle \geq c_i \left\langle v_{\pi(i+1)} - b_{\pi(i+1)} \right\rangle$$

By adding up these two inequalities, we get

$$(c_i - c_{i+1}) \left\langle v_{\pi(i)} - v_{\pi(i+1)} \right\rangle \geq 0, \text{ for all } i \in N \tag{4}$$

Edelman et al. (2007) derives this inequality, and incorrectly concludes the outcome to be efficient.

Lemma 1. *In any locally envy-free equilibrium of game* Γ^{GSP}*, advertiser* $1, 2, \ldots, k-1$ *win position* $1, 2, \ldots, k-1$*, respectively.*

Proof. We will first show that, in any Nash equilibrium, all advertisers $1, 2, \ldots, k-1$ win a position. Suppose, to the contrary, that there is an equilibrium $(b_1, b_2, \ldots, b_n)$ at which advertiser i doesn't win any position for some $i \in \{1, 2, \ldots, k-1\}$. Then there must exist at least two winning advertisers, say advertiser q and r, who value a click lower than advertiser i does. Without loss of generality, let $b_q > b_r$. Then we have $b_q > b_r \geq b_{\pi(k)} > b_{\pi(k+1)} \geq b_i$. Now it is easy to see that advertiser i can profitably increase his bid to a bid slightly higher than $b_{\pi(k)}$ so that he gets position k. Price he would pay is not higher than the price advertiser q paid before his deviation, which must be not greater than v_q. Therefore, this deviation of advertiser i is profitable - a contradiction.

Next, let $(b_1, b_2, \ldots, b_n)$ be a locally envy-free equilibrium. Then, by inequality (4), we must have $\pi(i) < \pi(i+1)$ for all $i \leq k$. Since advertiser 1 wins a position, it must be the case that $\pi(1) = 1$. Repeating this argument for advertisers $2, 3, \ldots, k-1$ sequentially, we get the result as claimed. □

We now understand that the locally envy-free condition alone is not sufficient to get such results. What would be the minimalistic and realistic assumption, in addition to locally envy-free condition, to get the desired results?

Definition 2. *An equilibrium bid in* Γ^{GSP} *is discontent-free, if a losing advertiser can not improve his payoff by getting the position and payment of the advertiser who wins the last position. That is, for all* $i > k$*,*

$$c_k v_{\pi(i)} - P_k^{GSP} \leq 0 \tag{5}$$

Theorem 1. *The outcome of any discontent-free and locally envy-free equilibrium in* Γ^{GSP} *is efficient, and the total revenue of the seller in it is at least as high as in the truthful VCG outcome.*

Proof. Take a discontent-free and locally envy-free equilibrium $(b_1, b_2, \ldots, b_n)$. By Lemma 1, advertiser i wins position i for all $i \leq k-1$. If advertiser k doesn't win a position, it must be $p_k \geq v_k$. Then advertiser $\pi(k)$ would be better off lowering his bid. Hence the outcome of this equilibrium is efficient.

We complete the proof by showing that P_i^{GSP}'s satisfy conditions (R1) and (R2). Note that, after having the efficiency, discontent free condition is equivalent to (R1), and locally envy-free condition is equivalent to (R2). □

3 Generalized Third-price Auctions

In GTP, the advertiser with the highest bid wins the first position, but pays the third-highest bid per click; the advertiser with the second-highest bid wins the second position, but pays the fourth-highest bid per click; and so on. Ties are broken uniformly at random. Given a bid profile $(t_1, t_2, \ldots, t_n)$, the payoff of advertiser $\pi(i)$, who wins position i, is simply $c_i \langle v_{\pi(i)} - t_{\pi(i+2)} \rangle$, where $t_{\pi(n+1)} = t_{\pi(n+2)} = 0$. We denote this game by Γ^{GTP}.

A bid profile $(t_1, t_2, \ldots, t_n)$ constitutes a Nash equilibrium of Γ^{GTP}, if

$$\begin{aligned} c_i \big\langle v_{\pi(i)} - t_{\pi(i+2)} \big\rangle &\geq c_j \big\langle v_{\pi(i)} - t_{\pi(j+2)} \big\rangle, \quad \forall i, j \in N : j \geq i - 1 \\ c_i \big\langle v_{\pi(i)} - t_{\pi(i+2)} \big\rangle &\geq c_j \big\langle v_{\pi(i)} - t_{\pi(j+1)} \big\rangle, \quad \forall i, j \in N : j < i - 1 \end{aligned} \tag{6}$$

Remember, in GSP, if advertiser $\pi(i)$ who wins position i wants to move up by one position, he would bid just above the bid of advertiser $\pi(i-1)$ and pays $b_{\pi(i-1)}$, the *bid* of the advertiser who is currently occupying position $i-1$. In contrast, in GTP, if advertiser $\pi(i)$ who wins position i wants to move up by one position, he still would bid just above the bid of advertiser $\pi(i-1)$ and pays $t_{\pi(i+1)}$, the *price* that advertiser $\pi(i-1)$ is currently paying.

Lemma 2. *The set of Nash equilibrium outcomes in Γ^{GTP} coincides with the set of locally envy-free equilibrium outcomes in Γ^{GSP}.*

Proof. Take a locally envy-free equilibrium $(b_1, b_2, \ldots, b_n)$ in Γ^{GSP}, and determine the permutation π of players according to this bid profile. Note that, by combining conditions (2) and (3), a bid profile $(b_1, b_2, \ldots, b_n)$ is a locally envy-free equilibrium in Γ^{GSP}, if and only if:

$$\begin{aligned} c_i \big\langle v_{\pi(i)} - b_{\pi(i+1)} \big\rangle &\geq c_j \big\langle v_{\pi(i)} - b_{\pi(j+1)} \big\rangle, \quad \forall i, j \in N : j \geq i - 1 \\ c_i \big\langle v_{\pi(i)} - b_{\pi(i+1)} \big\rangle &\geq c_j \big\langle v_{\pi(i)} - b_{\pi(j)} \big\rangle, \quad \forall i, j \in N : j < i - 1 \end{aligned} \tag{7}$$

Next, construct a bid profile $(t_1, t_2, \ldots, t_n)$ in Γ^{GTP} as $t_{\pi(i)} \equiv b_{\pi(i-1)}$ for all $i > 1$ and choose any value greater than $b_{\pi(1)}$ for $t_{\pi(1)}$. Then the resulting outcome of $(t_1, t_2, \ldots, t_n)$ in Γ^{GTP} is the same as the outcome of the locally envy-free equilibrium $(b_1, b_2, \ldots, b_n)$ in Γ^{GSP}. To see that $(t_1, t_2, \ldots, t_n)$ constitutes a Nash equilibrium in Γ^{GTP}, rewriting condition (7) in terms of t_i's gives us the Nash equilibrium condition (6) in Γ^{GTP}. Showing other direction is similar, thus omitted. □

Corollary 1. *In any Nash equilibrium of game Γ^{GTP}, advertiser $1, 2, \ldots, k-1$ win position $1, 2, \ldots, k-1$, respectively.*

Proof. It follows from Lemma 1 and Lemma 2. □

Definition 3. *An equilibrium bid in Γ^{GTP} is discontent-free, if a losing advertiser can not improve his payoff by getting the position and payment of the advertiser who wins the last position. That is, for all $i > k$,*

$$c_k v_{\pi(i)} - P_k^{GTP} \leq 0 \tag{8}$$

Theorem 2. *The outcome of any discontent-free equilibrium in Γ^{GTP} is efficient, and the total revenue of the seller in it is at least as high as in the truthful VCG outcome.*

Proof. Take a discontent-free equilibrium $(t_1, t_2, \ldots, t_n)$. By Corollary 1, advertiser i wins position i for all $i \leq k-1$. If advertiser k doesn't win a position, it must be $p_k \geq v_k$. Then advertiser $\pi(k)$ would be better off lowering his bid. Hence the outcome of this equilibrium is efficient.

We complete the proof by showing that P_i^{GTP}'s satisfy conditions (R1) and (R2). Note that, after having the efficiency, discontent free condition is equivalent to (R1), and the Nash equilibrium condition for $j = i-1$ in (6) implies (R2). □

4 Equilibrium Selection: Sensibility

Definition 4. *An equilibrium bid in Γ^{GTP} (or, Γ^{GSP}) is sensible, if losing advertiser bids at least his own value.*

Theorem 3. *The outcome of any sensible equilibrium in Γ^{GTP} is efficient, and the total revenue of the seller in it is at least as high as in the truthful VCG outcome.*

Proof. First we prove the efficiency. Since, by Corollary 1, advertiser i wins position i for all $i \leq k-1$, we just need to show that advertiser k wins position k in any sensible equilibrium. Suppose, to the contrary, that there exists a sensible equilibrium $(t_1, t_2, \ldots, t_n)$ where advertiser k doesn't win a position. Then advertiser $\pi(k)$ wins position k and pays $t_{\pi(k+2)}$ which implies that $v_{\pi(k)} \geq t_{\pi(k+2)}$. Since advertiser k is a losing advertiser, he bids at least his own value. Thus, $t_{\pi(k)} > t_k \geq v_k > v_{\pi(k)} \geq t_{\pi(k+2)}$. This implies $t_k = t_{\pi(k+1)}$, but then advertiser k would be better off moving one position up since $v_k > t_{\pi(k+2)}$ - a contradiction.

Now take a sensible equilibrium $(t_1, t_2, \ldots, t_n)$ in Γ^{GTP}. Then, as we argued in the proof of Theorem 2, condition (R2) is satisfied. We now just need to show that condition (R1) is satisfied, that is $P_k^{GTP} \geq c_k v_{k+1}$. Suppose this is not true, i.e., $t_{\pi(k+2)} < v_{k+1}$. Since, by sensibility, $v_{k+1} \leq t_{k+1}$, advertiser $k+1$ is the first one to be excluded from the winners list. Therefore, if advertiser $k+1$ bids slightly above t_k, he would get position k and makes positive profit as his payment would still be $t_{\pi(k+2)}$ - a contradiction. This concludes the proof. □

Theorem 4. *The outcome of any sensible and locally envy-free equilibrium in Γ^{GSP} is efficient, and the total revenue of the seller in it is at least as high as in the truthful VCG outcome.*

Proof. Similar, thus omitted. □

5 Impossibility and Possibility

Theorem 5. *There is no auction mechanism M such that*

1. $W^M(b) \geq W^{VCG}$
2. $R^M(b) \geq R^{VCG}$

for any Nash equilibrium b of the game induced by M.

Proof. ... TO BE COMPLETED ... □

Theorem 6. *For any $\varepsilon > 0$, there is an auction mechanism M_ε such that*

1. $W^{M_\varepsilon}(b) \geq (1-\varepsilon)W^{VCG}$
2. $R^{M_\varepsilon}(b) \geq (1-\varepsilon)R^{VCG}$

for any Nash equilibrium b of the game induced by M_ε.

Proof. The proof is constructive. For a given $\varepsilon > 0$, consider following auction mechanism that we call Randomized Third-price Auction (RTP), denoted by $\Gamma^{RTP}(\varepsilon)$. The payment in RTP is the same as in GTP, that is if an advertiser wins a position he pays the bid of the advertiser ranked two position below him. Moreover, the allocation in RTP is the same as in GTP except the last position. That is, the highest bidder wins the first position, and the second highest bidder wins the second position, and so on. However, k-th highest bidder wins position k with probability $1-\varepsilon$, and $(k+1)$-st highest bidder wins position k with the remaining probability. For a bid profile $(t_1, t_2, \ldots, t_n)$, we denote by $\pi(i)$ the identity of the advertiser who bids the i-th highest bid. Then the payoff of advertiser $\pi(i)$ can be written as $\gamma_i\langle v_{\pi(i)} - t_{\pi(i+2)}\rangle$, where

$$\gamma_i = \begin{cases} c_i, & i \neq k, k+1 \\ (1-\varepsilon)c_i, & i = k \\ \varepsilon c_i, & i = k+1 \end{cases}$$

Game $\Gamma^{RTP}(\varepsilon)$ is therefore strategically equivalent to $\Gamma^{GTP}(\gamma)$ - a GTP game with click-through-rates $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$, in which $k+1$ positions have positive click-through-rate.

Now let $(t_1, t_2, \ldots, t_n)$ be a Nash equilibrium in $\Gamma^{GTP}(\gamma)$. Then, by Corollary 1, we have $\pi(i) = i$ for all $i \leq k$, i.e,

$$t_1 > t_2 > \ldots > t_{k-1} > t_k > t_{\pi(k+1)} > t_{\pi(k+2)} \geq \ldots \geq t_{\pi(n)}$$

Consequently, we get

$$\begin{aligned} W^{GTP(\gamma)}(t) &= \sum_{i=1}^{k+1} \gamma_i v_{\pi(i)} = \sum_{i=1}^{k} \gamma_i v_i + \gamma_{k+1} v_{\pi(k+1)} \\ &\geq \sum_{i=1}^{k} \gamma_i v_i \geq (1-\varepsilon) \sum_{i=1}^{k} c_i v_i = (1-\varepsilon) W^{VCG} \end{aligned}$$

Next, consider following two cases: $\pi(k+1) \neq k+1$ and $\pi(k+1) = k+1$. In the first case, it must be $t_{\pi(k+2)} \geq v_{k+1}$, otherwise advertiser $k+1$ can improve his payoff by outbidding advertiser $\pi(k+1)$. This yields $P_k^{GTP(\gamma)} = \gamma_k t_{\pi(k+2)} \geq (1-\varepsilon) c_k v_{k+1}$. In the second case, advertiser $k+1$ wins position $k+1$ and he can not improve his payoff by outbidding advertiser k, that is,

$$\gamma_{k+1} v_{k+1} - P_{k+1}^{GTP(\gamma)} \geq \gamma_k v_{k+1} - P_k^{GTP(\gamma)}$$

which implies $P_k^{GTP(\gamma)} \geq (1-2\varepsilon) c_k v_{k+1}$. So, in both cases, we have

$$P_k^{GTP(\gamma)} \geq (1-2\varepsilon) c_k v_{k+1} \tag{9}$$

Moreover, for $i < k$, by the Nash equilibrium condition that advertiser $i+1$ is not better off outbidding advertiser i, we get

$$P_i^{GTP(\gamma)} \geq (1-2\varepsilon)(c_i - c_{i+1}) v_{i+1} + P_{i+1}^{GTP(\gamma)} \tag{10}$$

Now using induction, we get $P_i^{GTP(\gamma)} \geq (1-2\varepsilon) P_i^{VCG}$ for all i. Therefore,

$$R^{GTP(\gamma)}(t) \geq (1-2\varepsilon) R^{VCG}$$

To be precise, game $\Gamma^{RTP}(\varepsilon/2)$ satisfies the two conditions stated in the theorem. This completes the proof. □

References

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